Pre-class Warm-up!!!

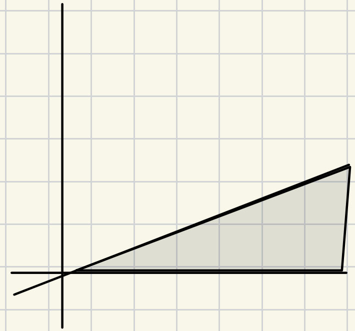
Which picture shows the region of integration of the double integral

$$\int_0^1 \int_{2x}^2 e^{\sin(xy)}\, dy\, dx$$

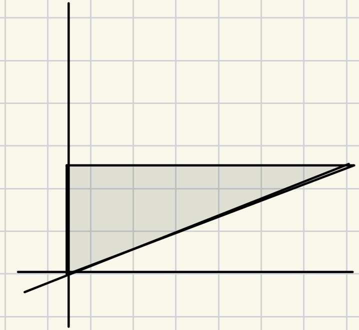
The lines $y=2x$ and $y=2$ are significant

If you have to sketch a region of integration, show features like the drawings on the right: axes, lines approximately going through the origin, labeling if needed to clarify.

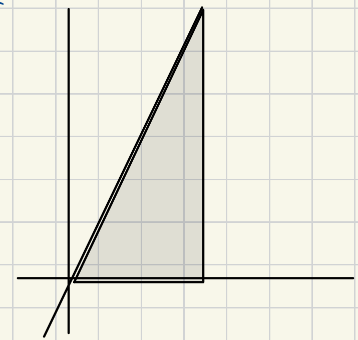
a.



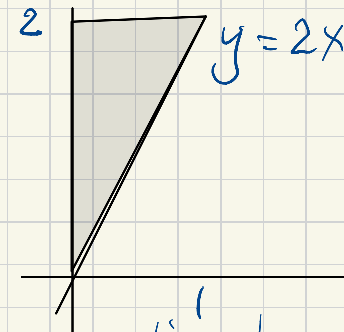
b.



c.



d. ✓



y lies above 2x, below 2.

e. None of the above.

Section 5.5 The Triple Integral

We learn

- The triple integral over a rectangular box is defined by Riemann sums.
- The integral satisfies similar formal properties to the double integral.
- We can extend the triple integral to elementary regions in R^3.
- We get practice doing triple integrals over more complicated regions.

Typical questions:

a. Like 5.5 questions 3, 4: Perform the integration over the box indicated

$$\iiint_B xy \, dx\, dy\, dz \quad \text{where } B = [0,1]\times[1,2]\times[1,2]$$

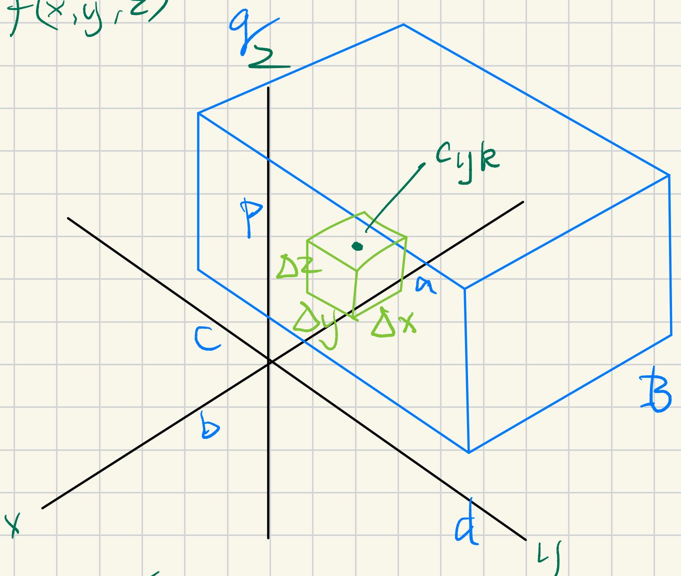
or $\iiint_B xy \, dV$. or $\int_1^2 \int_{-1}^{2} \int_0^1 xy \, dx\, dy\, dz$

b. Like 5.5 questions 11, 12: Find the volume of the region bounded by the surfaces

z = 2y, z = 6-y, y = x^2, y = 2-x^2

Riemann sums

These are similar to the 2-D case. We have a rectangular box
$B = [a,b] \times [c,d] \times [p.q]$ and a function $f(x,y,z)$



We divide each side of the box into small intervals of lengths $\Delta x, \Delta y, \Delta z$

A Riemann sum looks like

$$\sum_{i,j,k} f(c_{ijk})\,\Delta x\,\Delta y\,\Delta z$$

We say f is integrable over B if these sums approach a common value d as $\Delta x, \Delta y, \Delta z \to 0$, c_{ijk} takes on all possibilities.

Notation: $d = \iiint_B f\,dV = \iiint_B f\,dx\,dy\,dz$

$$= \int_p^q \int_c^d \int_a^b f\,dx\,dy\,dz$$

- The integral of f is number.
- We can interpret $\iiint F\, dV$ in various ways:

1. $\iiint_B 1\, dV$ = 3-D volume of B

2. $\iiint_B f\, dV$ = 4-D volume under the graph of f in $\mathbb{R}^4$.

Properties:

- Integrals over a box can be evaluated as iterated integrals.
- Fubini's theorem
- Things like the upper and lower bounds, the mean value theorem

functions

$$\iiint af + bg \, dV = a \iiint f \, dV + b \iiint g \, dV$$

numbers

Typical questions:

a. Like 5.5 questions 3, 4: Perform the integration over the box indicated

$$\iiint_B xy \; dx \, dy \, dz \quad \text{where } B = [0,1] \times [1,2] \times [1,2]$$

or $\iiint_B xy \, dV$.

The integral is $\int_1^2 \int_{-1}^2 \int_0^1 xy \, dx \, dy \, dz$

$$= \int_1^2 \int_{-1}^2 \left[\frac{x^2 y}{2} \right]_0^1 dy \, dz$$

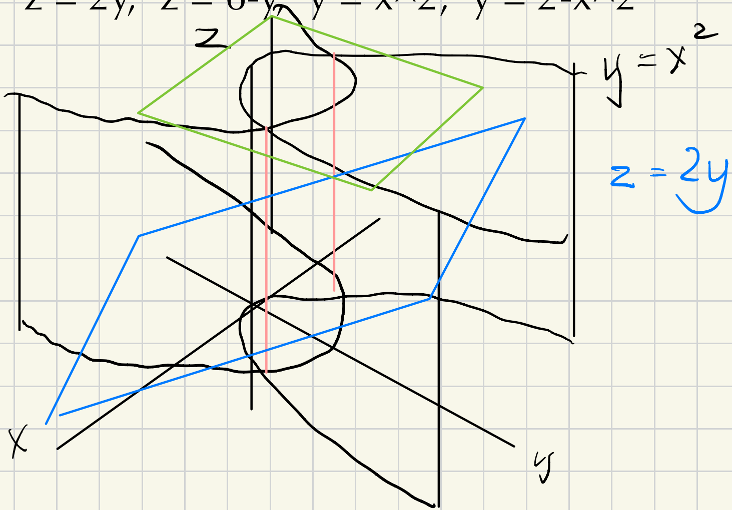
$$= \int_1^2 \int_{-1}^2 \frac{y}{2} \, dy \, dz = \int_1^2 \left[\frac{y^2}{4} \right]_{-1}^2 dz$$

$$= \int_1^2 \left(1 - \frac{1}{4}\right) dz = \left[\frac{3z}{4} \right]_1^2$$

$$= \frac{3}{4}$$

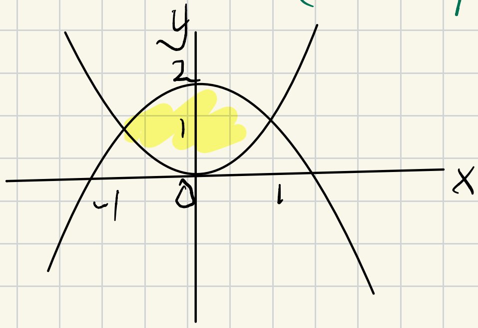
Elementary regions

b. Like 5.5 questions 11, 12: Find the volume of the region bounded by the surfaces
$z = 2y$, $z = 6-y$, $y = x^2$, $y = 2-x^2$



This region extends vertically with cut off ends and

cross-section (most of the time)



We integrate w.r.t z first

$$\int_{-1}^{1}\int_{x^2}^{2-x^2}\int_{2y}^{6-y} dz\,dy\,dx =$$

Pre-class Warm-up!!!

Answer: (i) (ii) (iii) (iv)

Announcement: Mathematica Labs will due on Wednesday 11:59pm in future.

Match integral (a) to the correct region of integration

y goes between $y=-\sqrt{9-x}$ and $y=\sqrt{9-x}$

$x = 9-y^2$

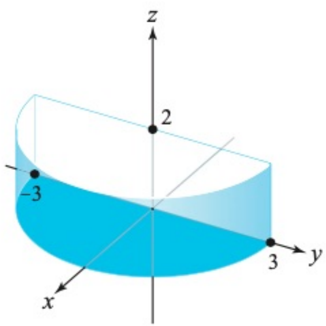
1. In parts (a) through (d) below, each iterated integral is an integral over a region D. Match the integral with the correct region of integration.

(a) $\int_0^2 \int_0^3 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dz\,dx$

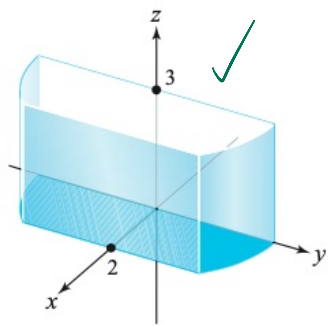
(b) $\int_0^2 \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz$

(c) $\int_0^1 \int_0^x \int_0^y dz\,dy\,dx$

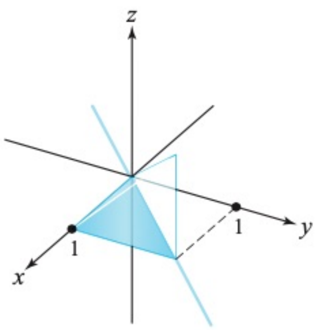
(d) $\int_0^1 \int_0^y \int_0^x dz\,dx\,dy$



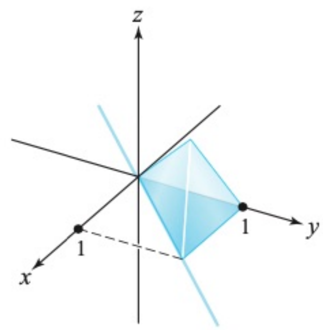
(i)



(ii)



(iii)



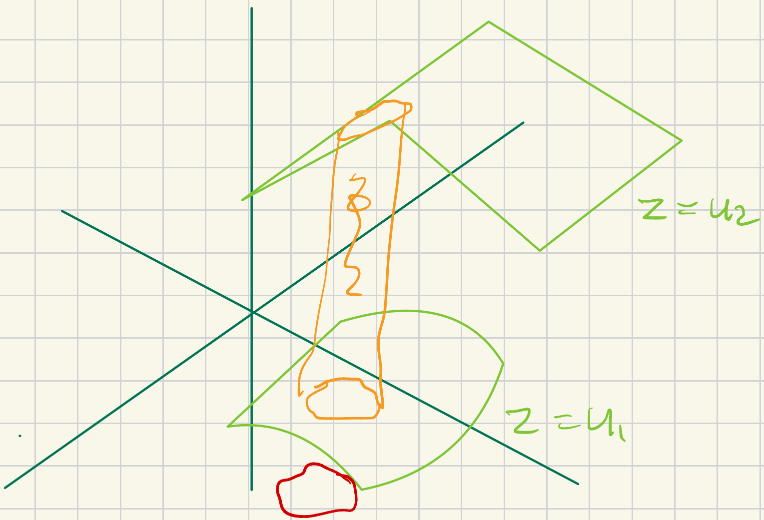
(iv)

2. Evaluate the following triple integral:

$$\iiint_W \sin x\,dx\,dy\,dz,$$

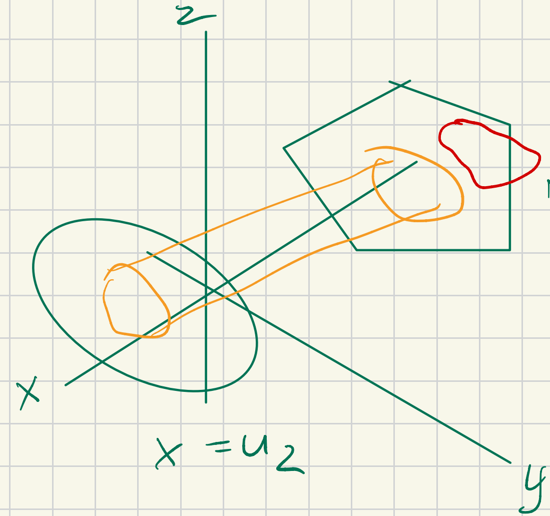
where W is the solid given by $0 \le x \le \pi$, $0 \le y \le 1$, and $0 \le z \le x$.

Elementary regions are regions between the graphs of two functions like $z = u_1(x,y)$, $z = u_2(x,y)$



This is z-simple.

or $x = u_1(y,z)$ $u_2(y,z)$



This is x-simple

Elementary regions.

In question 1, which of (i), (ii), (iii), (iv) are elementary?
With respect to which of x, y, and z?

Which variable would you use to integrate first in (i), (ii) ?

Are any of (i) (ii) (iii) (iv) not elementary w.r.t. one of x, y, z ?

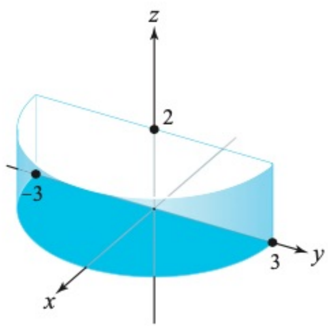
Yes

No

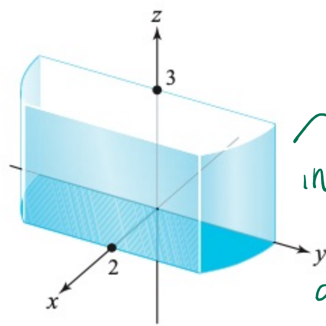
1. In parts (a) through (d) below, each iterated integral is an integral over a region D. Match the integral with the correct region of integration.

(a) $\displaystyle\int_0^2\int_0^3\int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dz\,dx$ (c) $\displaystyle\int_0^1\int_0^x\int_0^y dz\,dy\,dx$

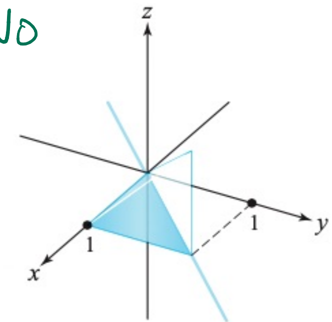
(b) $\displaystyle\int_0^2\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz$ (d) $\displaystyle\int_0^1\int_0^y\int_0^x dz\,dx\,dy$



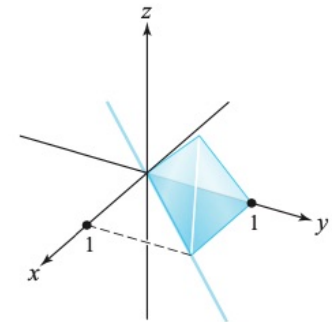
(i)



(ii)



(iii)



(iv)

2. Evaluate the following triple integral:

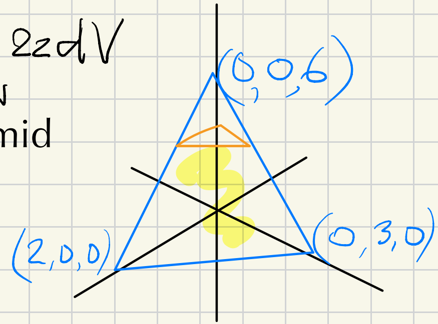
$$\iiint_W \sin x \, dx\, dy\, dz,$$

where W is the solid given by $0 \le x \le \pi$, $0 \le y \le 1$, and $0 \le z \le x$.

integrating wrt x first is a bad choice

Example. Find $\iiint_W 2z\,dV$

where W is the pyramid



Solution Get an equation for the blue plane

$z = Ax + By + D$ where A = slope in x-direction $= -3$

B = slope in y-direction $= -2$

$z = 6 - 3x - 2y$

The integral is:

$$\int_0^2 \int_0^{3-3x/2} \int_0^{6-3x-2y} 2z\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^{2-\frac{2y}{3}} \int_0^{6-3x-2y} 2z\,dz\,dx\,dy$$

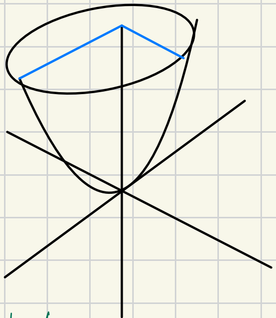
The blue plane is z = 6-3x-2y

The bottom edge is y = 3 - 3x/2

Answ. 9

Example. Find $\iiint_W 3xy\,dV$

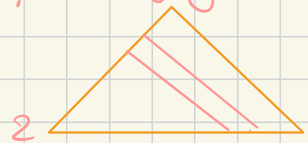
where W is the region bounded by the plane z = 2, and the surface z = x^2 + 2y^2 in the region x > 0, y > 0



It is

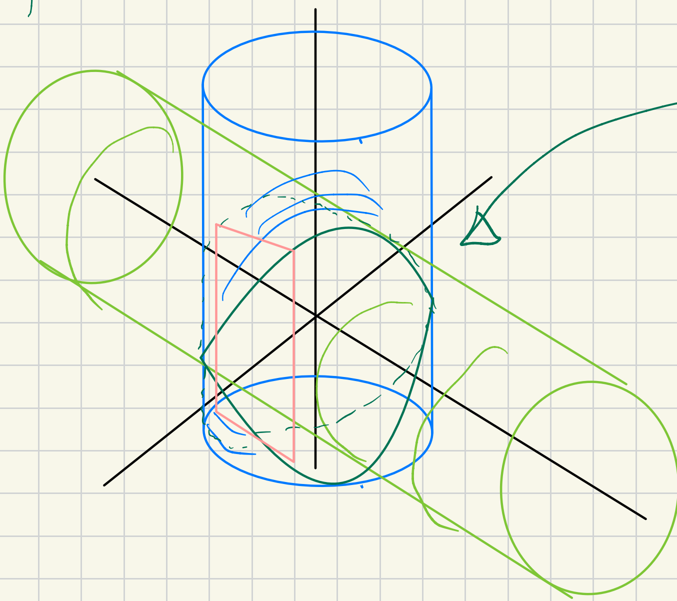
$$\int_0^{\sqrt{2}} \int_0^{\sqrt{1-\frac{x}{2}}} \int_{x^2+2y^2}^{2} 3xy\,dz\,dy\,dx$$

first integral,

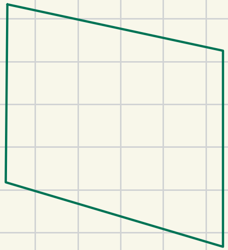


Answ: $\frac{1}{2}$

Like question 14.
Which variable would you choose to integrate with first? The region is the intersection of two cylinders of radius a



View from the x-axis. The cross-section is square

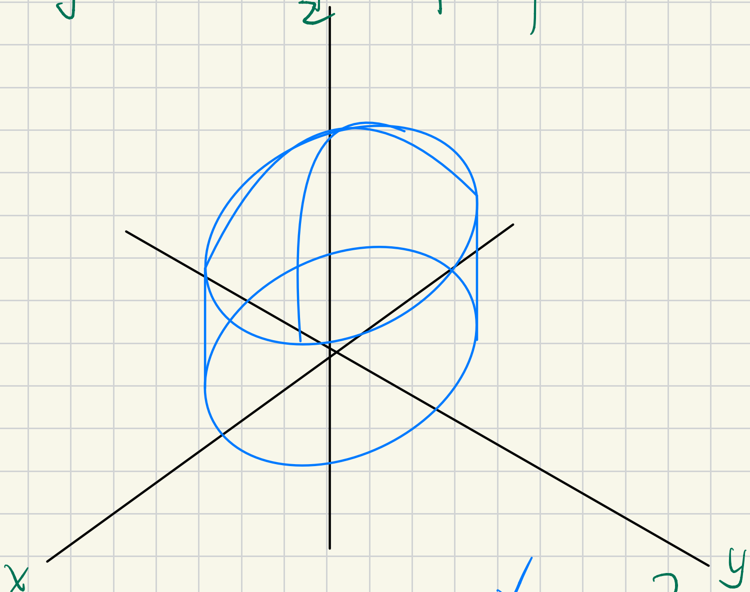


Integrate first w.r.t. y and z.
Last x.
We get

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} f \, dy\,dz\,dx$$

Like question 27. Which variable would you use to integrate first?

Cylinder with part of a sphere on top



x y z ✓ ?

x, y first: breaks the $\int$ in 2 pieces

Sphere top requires

$x^2 + y^2 + z^2 \leq b$

$$\int_{-a}^{a} \int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}} \int_{0}^{\sqrt{b-x^2-y^2}} dz \; dy \; dx$$

where the cylinder's $x^2+y^2 \leq a$

Also $z \geq 0$.